Lem 5 Suppose that $r>0$

$$\Pi_{2/1} := \mathrm{Ker}\left(\Pi_2 \xrightarrow{\mathrm{pr}_1} \Pi_1\right)$$

Then the kernel of the natural action

$$\rho : \Pi_1 \longrightarrow \mathrm{Aut}_{\mathbb{Z}_\ell}\left(\Pi_{2/1}^{ab}\right)$$

coincides with the kernel of the natural surj

$$\Pi_1 \twoheadrightarrow \left(\overline{\Pi}_1\right)^{ab}$$

— where $\overline{\Pi}_1 := \Pi_{X^{cpt}}$ — induced by

$$X \underset{\text{open}}{\hookrightarrow} \underbrace{X^{cpt}}_{\text{the smooth proper compactification of } X}$$

ernel of

rj

— induced by

$X \overset{\text{open}}{\hookrightarrow} X^{cpt}$

the smooth proper compactification of X

Since the action ρ of Π_1 on $\Pi_{2/1}^{ab}$

preserves the ext seq

$$0 \longrightarrow \mathbb{Z}_\ell(1) \longrightarrow \Pi_{2/1}^{ab} \longrightarrow \Pi_1^{ab} \longrightarrow 0$$

$[X_2 \hookrightarrow X \times_k X]$

[cf. the well-known str of a surface gp]

and induces identity automorphisms

on $\mathbb{Z}_\ell(1)$, Π_1^{ab}, we have

$$\begin{tikzcd}
\Pi_1 \arrow[r, "\rho"] \arrow[dr, "\phi"'] & \mathrm{Aut}(\Pi_{2/1}^{ab}) \\
 & \mathrm{Hom}(\Pi_1^{ab}, \mathbb{Z}_\ell(1)) \arrow[u] \arrow[ul, phantom, "\circlearrowright"]
\end{tikzcd}$$

$\Rightarrow$ We want to compute

$\ker\phi$

$\delta_X \hookrightarrow X \times_k X$: the diag divisor

$\mathcal{L}$: the invertible sheaf corr to δ_X (on $X \times_k X$)

$\mathbb{L}^\times \to X \times_k X$: the complement of the zero section of the geom. line bundle det by $\mathcal{L}$

$\uparrow$
$\mathbb{G}_m$-torsor

In particular,

$\mathcal{O}_{X \times_k X} \hookrightarrow \mathcal{L}$ induces

$$\begin{tikzcd} X_2 \arrow[rr] \arrow[dr] & \wr & \mathbb{L}^\times \arrow[dl] \\ & X \times_k X & \end{tikzcd}$$

$\rightsquigarrow$

$$\begin{tikzcd} 1 \arrow[r] & \mathrm{Ker} \arrow[r] \arrow[d] & \Pi_2 \arrow[r] \arrow[d] & \Pi_1 \times \Pi_1 \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & \mathbb{Z}_\ell(1) \arrow[r] & \Pi_{\mathbb{L}^\times} \arrow[r] & \Pi_1 \times \Pi_1 \arrow[r] & 1 \end{tikzcd}$$

corresponding $\to$ ☆

Then, it follo

vari

Serre-

rel

$H^2(\Pi_1 \times \Pi_1, \mathbb{Z}_\ell(1$

☆ $\mapsfrom$

icular,

$\hookrightarrow \mathcal{L}$ induces

$$\begin{tikzcd}
X_2 \arrow[rr] \arrow[dr] & \arrow[d, phantom, "\circlearrowright"] & \mathbb{L}^\times \arrow[dl] \\
& X \times_k X &
\end{tikzcd}$$

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{Ker} \arrow[r] \arrow[d] & \Pi_2 \arrow[r] \arrow[d] & \Pi_1 \times \Pi_1 \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \mathbb{Z}_\ell(1) \arrow[r] & \Pi_{\mathbb{L}^\times} \arrow[r] & \Pi_1 \times \Pi_1 \arrow[r] & 1
\end{tikzcd}$$

corresponding $\circledast$

Then, it follows from

various definitions

involved

that

Serre-Hochschild rel to $\Pi_1 \times \Pi_1 \overset{p_1}{\twoheadrightarrow} \Pi$

$$\begin{tikzcd}
H^2(\Pi_1 \times \Pi_1, \mathbb{Z}_\ell(1)) \arrow[r, "\sim"] & H^1(\Pi_1, \mathrm{Hom}(\Pi_1^{ab}, \mathbb{Z}_\ell(1))) \arrow[d, "\wr"] \\
\circledast & \mathrm{Hom}(\Pi_1, \mathrm{Hom}(\Pi_1^{ab}, \mathbb{Z}_\ell(1))) \\
& \phi \arrow[u, phantom, sloped, "\in"] \arrow[ul, bend left, mapsto]
\end{tikzcd}$$

On the other hand, since

$$\begin{tikzcd}
H^2(\Pi_1 \times \Pi_1, \mathbb{Z}_\ell(1)) \arrow[r, phantom, "\ni"] & \circledast & \\
H^2_{et}(X \times_k X, \mathbb{Z}_\ell(1)) \arrow[u, "\wr"] & & H^2_{et}(X^{pt} \times_k X^{pt}, \mathbb{Z}_\ell(1)) \arrow[ll] \\
c_1(\delta_x) \arrow[u, phantom, sloped, "\in"] \arrow[uur, mapsto] & & c_1(\delta_{X^{pt}}) \arrow[ll, mapsto] \\
\text{1-st chern classes} \arrow[u] \arrow[urr] & &
\end{tikzcd}$$

we conclude that

ϕ coincides

$$\Pi_1 \twoheadrightarrow \overline{\Pi}_1^{ab} \xrightarrow{\sim} \mathrm{Hom}(\overline{\Pi}_1^{ab}, \mathbb{Z}_\ell(1))$$

$$\hookrightarrow \mathrm{Hom}(\Pi_1^{ab}, \mathbb{Z}_\ell(1))$$

Poincaré duality

[cf., e.g., [Milne], ch VI, lem 11.2]

$\therefore \ker \rho = \ker \phi$

$= \ker(\Pi_1 \twoheadrightarrow \overline{\Pi}_1^{ab})$ //

<u>Prop 6</u> $\exists \beta_2 : \Pi_2^{\circ} \xrightarrow{\sim} \Pi_2^{\bullet}$: PF-adm

Then $(g^{\circ}, r^{\circ}) = (g^{\bullet}, r^{\bullet})$

(∵) Let $p_i^{\square} : \Pi_2^{\square} \twoheadrightarrow \Pi_1^{\square}$ be the natural outer surj induced by

the i-

$\overline{\Pi}_1^{ab}$) //

$\Pi_2^\bullet$: PF-adm

) = ($g^\bullet, r^\bullet$)

$\Pi_1^{\square}$ be the

nj induced by

the i-th proj $X_2^\square \longrightarrow X^\square$

To verify $\boxed{\text{prop6}}$, we may assume WLOG that

$$\beta(\ker p_i^\circ) = \ker p_i^\bullet \quad (i=1,2)$$

$$\rightsquigarrow \begin{tikzcd} \Pi_2^\circ \arrow[r, "\sim"] \arrow[d, "p_i^\circ"] \arrow[dr, phantom, "\circlearrowright"] & \Pi_2^\bullet \arrow[d, "p_i^\bullet"] \\ \Pi_1^\circ \arrow[r, "\sim"] & \Pi_1^\bullet \end{tikzcd}$$

Here note that

$r^\circ > 0 \Longleftrightarrow \Pi_1^\circ$: free pro-ℓ

$\Longleftrightarrow \Pi_1^\bullet$: free pro-ℓ

$\Longleftrightarrow r^\bullet > 0$

Thus, if $r^\circ = 0$, then $r^\bullet = 0$

Moreover, in this case

$$2g^\circ = \mathrm{rk}_{\mathbb{Z}_\ell}(\Pi_1^\circ)^{ab} = \mathrm{rk}_{\mathbb{Z}_\ell}(\Pi_1^\bullet)^{ab} = 2g^\bullet$$

$\therefore\ g^\circ = g^\bullet$ //

So we may assume that

$r^\circ > 0 \quad (\Leftrightarrow r^\bullet > 0)$

Then

$$\begin{tikzcd}
1 \arrow[r] & \Pi_{2/1}^\circ \arrow[r] \arrow[d, "\wr"] & \Pi_2^\circ \arrow[r, "p_1^\circ"'] \arrow[d, "\wr\ \beta_2"] & \Pi_1^\circ \arrow[r] \arrow[d, "\wr"] & 1 \\
1 \arrow[r] & \Pi_{2/1}^\bullet \arrow[r] & \Pi_2^\bullet \arrow[r, "p_1^\bullet"'] & \Pi_1^\bullet \arrow[r] & 1
\end{tikzcd}$$

induces

$$\begin{tikzcd}
\Pi_1^\circ \arrow[r, "\rho^\circ"] \arrow[d, "\wr"] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Aut}(\Pi_{2/1}^\circ) \arrow[d, "\wr"] \\
\Pi_1^\bullet \arrow[r, "\rho^\bullet"'] & \mathrm{Aut}(\Pi_{2/1}^\bullet)
\end{tikzcd}$$

$\therefore\ \mathrm{Ker}(\rho^\circ) \xrightarrow{\sim} \mathrm{Ker}(\rho^\bullet)$

$\underset{\text{lem5}}{\Longrightarrow}\ \mathrm{Ker}(\Pi_1^\circ \twoheadrightarrow (\overline{\Pi}_1^\circ)^{ab})$

$\xrightarrow{\sim} \mathrm{Ker}(\Pi_1^\bullet \twoheadrightarrow (\overline{\Pi}_1^\bullet)^{ab})$

$\Rightarrow$

$\therefore\ ($

lem5 + some argument

theory of the wt filtration

of Π_n

$$\Rightarrow \ker\left((\Pi_1^{\bullet})^{ab} \twoheadrightarrow (\overline{\Pi}_1^{\circ})^{ab}\right)$$

$$\xrightarrow{\sim} \ker\left((\Pi_1^{\circ})^{ab} \twoheadrightarrow (\overline{\Pi}_1^{\bullet})^{ab}\right)$$

$$\therefore (2g^{\circ}+r^{\circ}+1)-2g^{\circ}$$

$$=(2g^{\bullet}+r^{\bullet}+1)-2g^{\bullet}$$

$$\therefore r^{\circ}=r^{\bullet}$$

$$\therefore g^{\circ}=g^{\bullet} \,/\!/$$

$\boxed{\text{Prop 7}}$ Recall that $n \geq 2$

Let $\varphi : \Pi_n^{ab} \to \underbrace{\Pi_1^{ab} \times \cdots \times \Pi_1^{ab}}_{n}$

be the surj induced by

$$X_n \hookrightarrow X \times_k \cdots \times_k X$$