

lem 5 Suppose that $r > 0$

$$\pi_{2,1} := \ker(\pi_2 \twoheadrightarrow \pi_1)$$

\swarrow
Pr 1

Then the kernel of the natural action

$$\rho: \pi_1 \longrightarrow \text{Aut}_{\pi_1}(\pi_{2,1}^{ab})$$

coincides with the kernel of the natural surj

$$\pi_1 \twoheadrightarrow (\overline{\pi_1})^{ab}$$

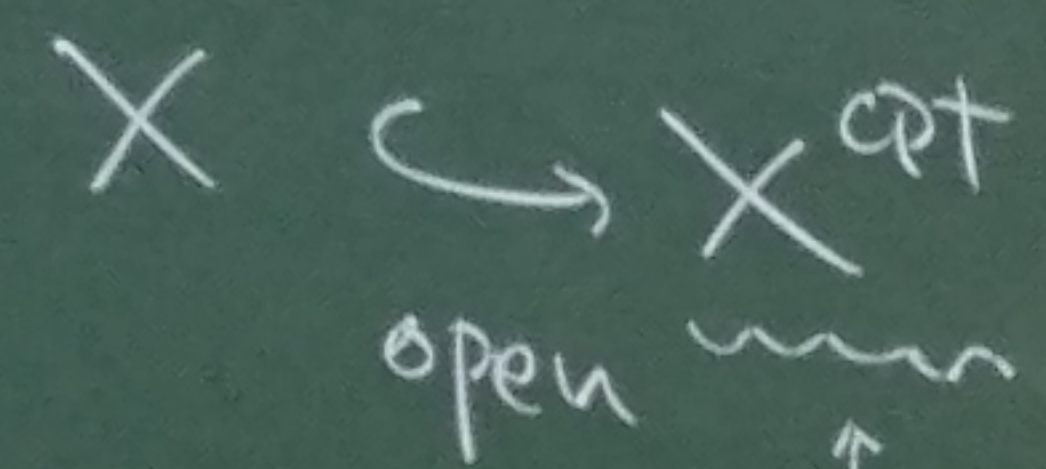
— where $\overline{\pi_1} := \pi_{X^{cpt}}$ — induced by

$$X \xrightarrow{\text{open}} X^{cpt}$$

the smooth proper compactification of X

kernel of
ij

induced by



the smooth proper
compactification
of X

Since the action ρ of
 π_1 on $\pi_{2,1}^{\text{ab}}$
preserves the ext seq

$$0 \rightarrow \mathbb{Z}\ell(1) \rightarrow \pi_{2,1}^{\text{ab}} \rightarrow \pi_1^{\text{ab}} \rightarrow 0$$

[$X_2 \hookrightarrow X \times_{\mathbb{R}} X$]

[cf. the well-known str of a surface gp]
and induces identity automorphisms

on $\mathbb{Z}\ell(1)$, π_1^{ab} , we have

$$\begin{array}{ccc}
 \pi_1 & \xrightarrow{\rho} & \text{Aut}(\pi_{2,1}^{\text{ab}}) \\
 \phi \searrow & ? & \nearrow \\
 & & \text{Hom}(\pi_1^{\text{ab}}, \mathbb{Z}\ell(1))
 \end{array}$$

\Rightarrow We want to compute
ker

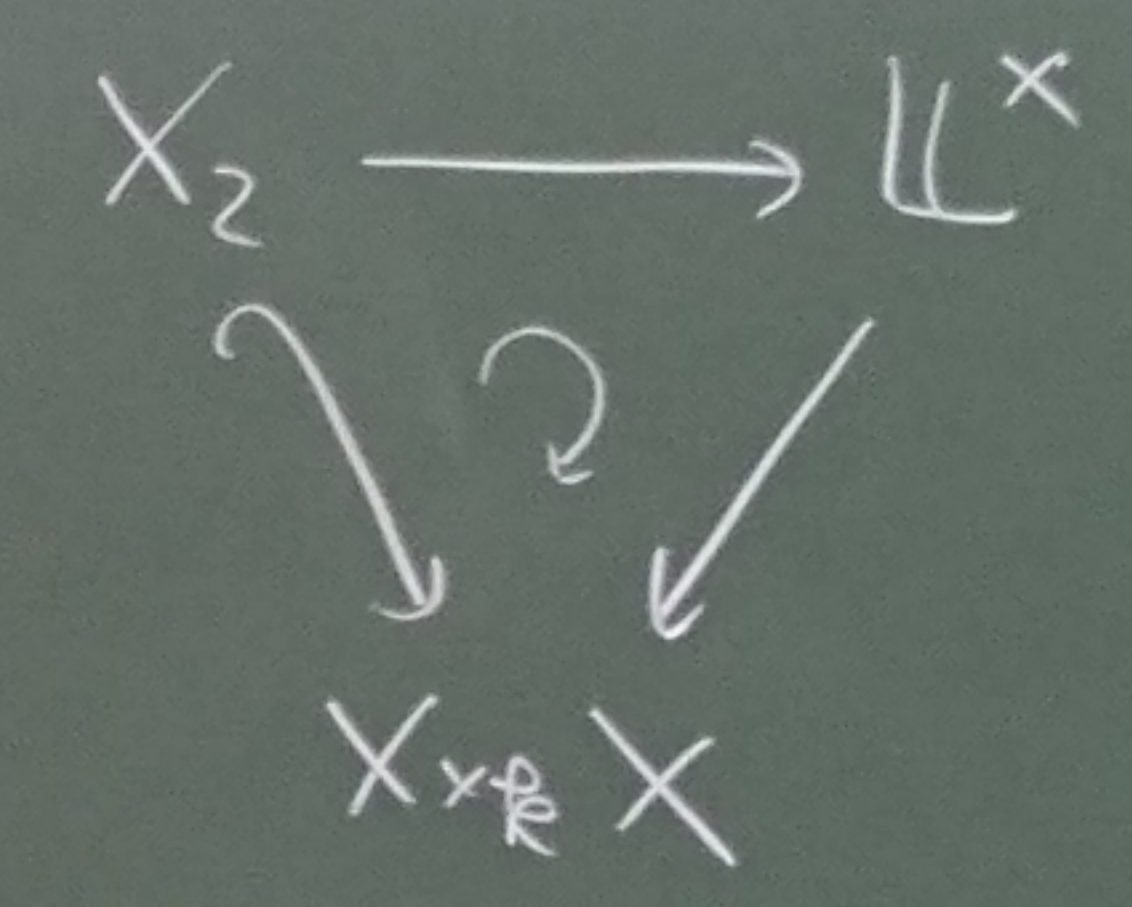
$\delta_x \hookrightarrow X \times_{\mathbb{P}^1} X$: the diag
divisor

\mathcal{L} : the invertible sheaf
corr to δ_x
(on $X \times_{\mathbb{P}^1} X$)

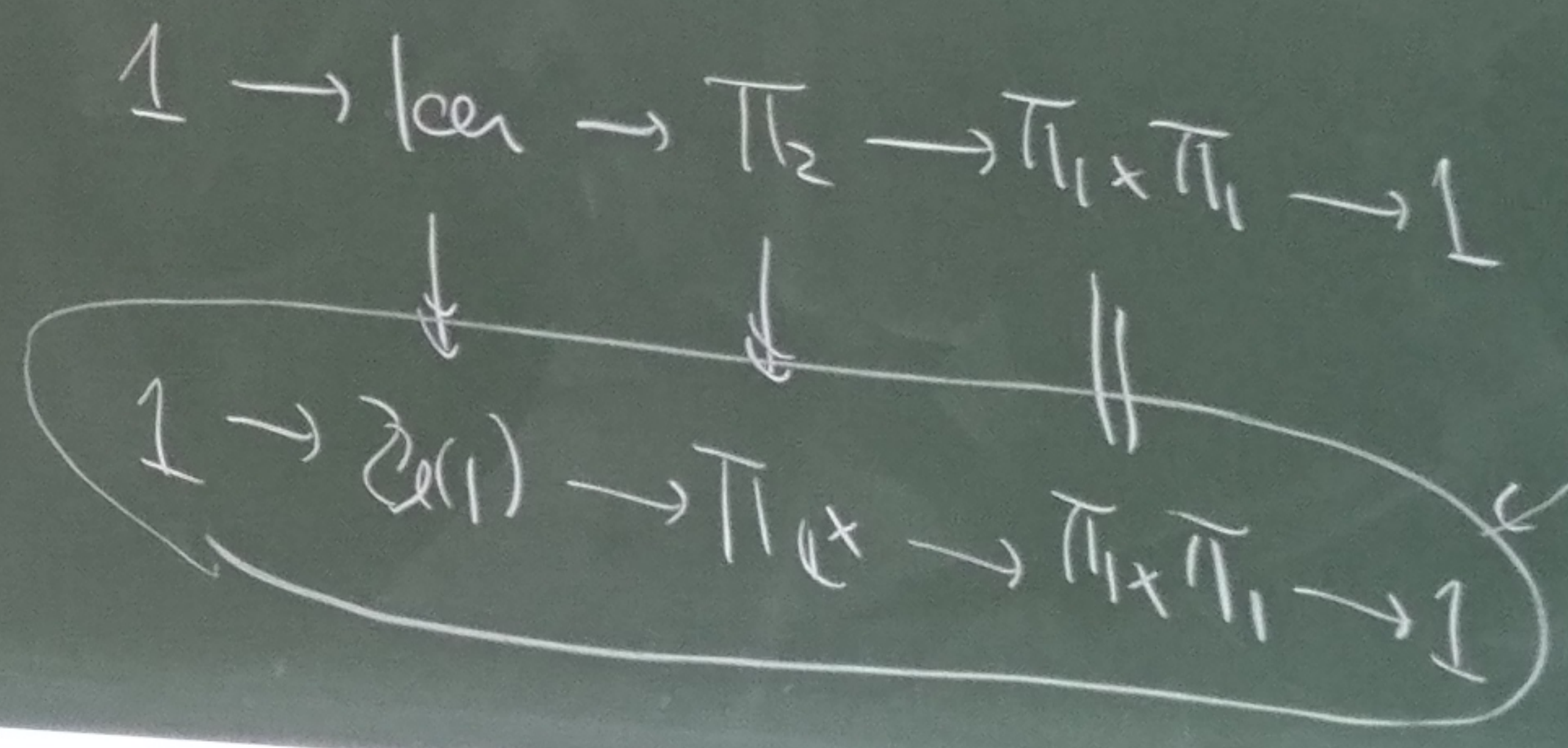
$\mathcal{L}^x \rightarrow X \times_{\mathbb{P}^1} X$: the complement
of the zero section
of the geom. line
bundle det by \mathcal{L}
↑
div-tensor

In particular,

$\mathcal{O}_{X \times_{\mathbb{P}^1} X} \hookrightarrow \mathcal{L}$ induces

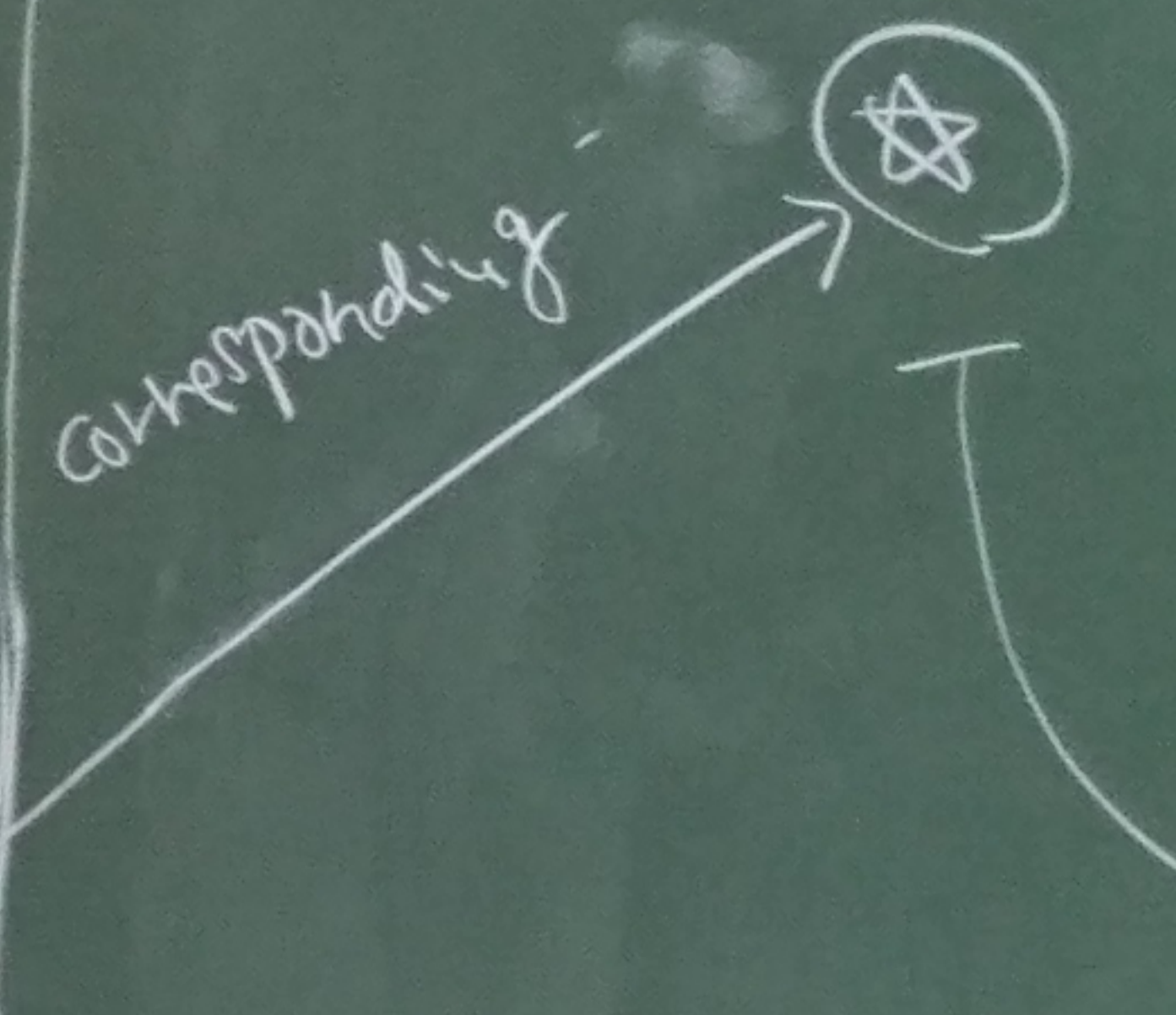


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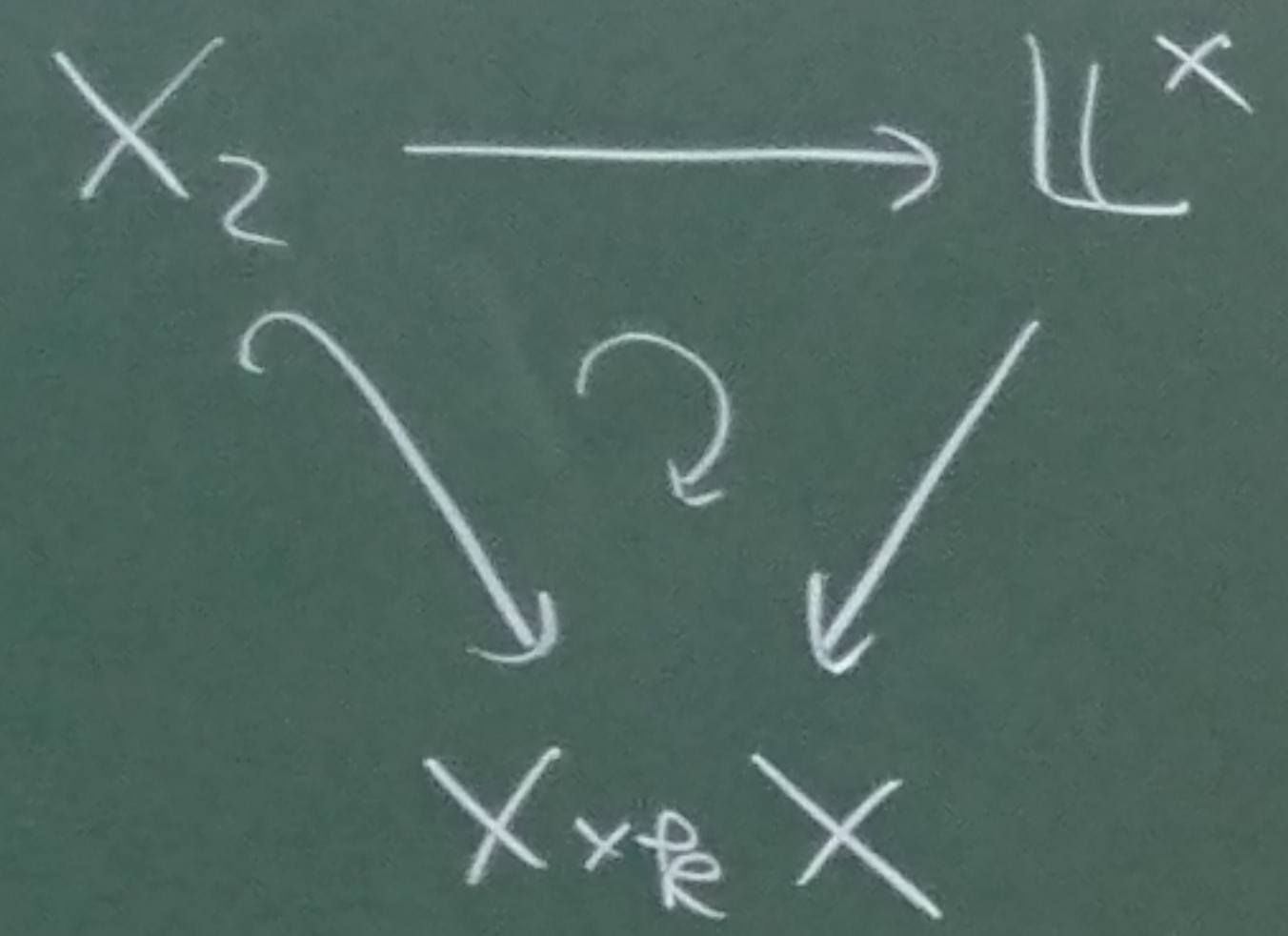


Then, it follows

$H^2(\pi_1 \times \pi_1, \mathbb{Z}(1))$



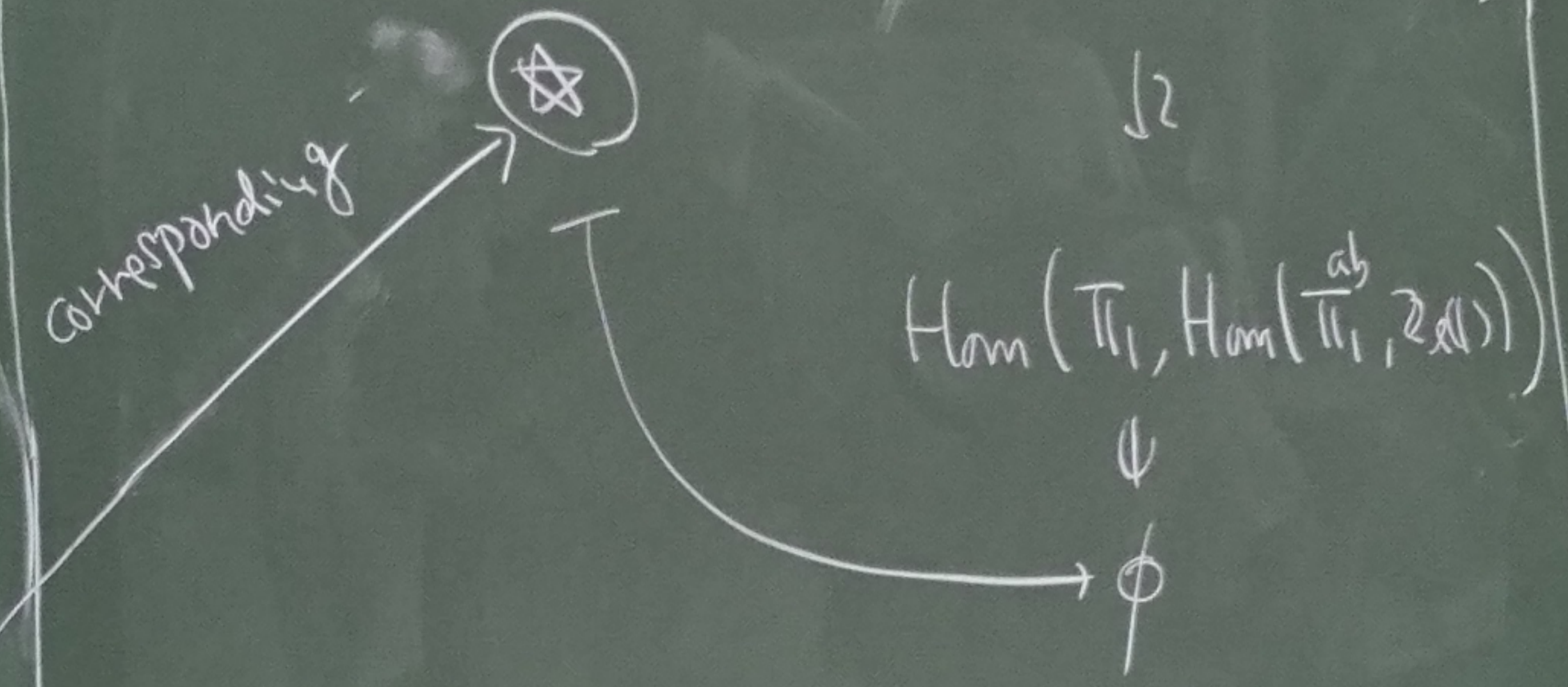
icular,  
 $\hookrightarrow \mathcal{L}$  induces



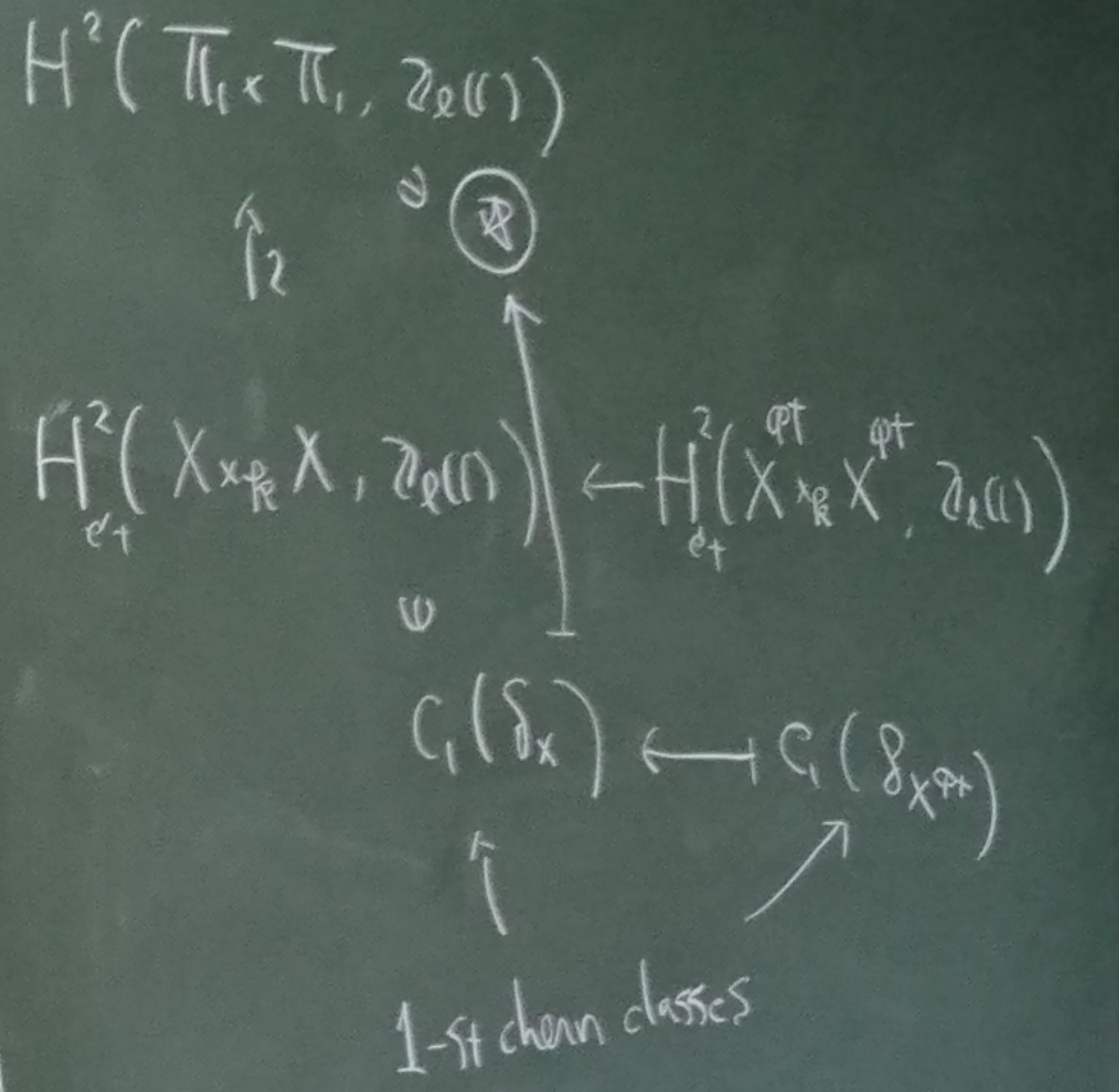
$$\begin{array}{ccccccc}
 1 & \longrightarrow & \ker & \longrightarrow & \pi_2 & \longrightarrow & \pi_1 \times \pi_1 \longrightarrow 1 \\
 & & \downarrow & & \downarrow & & \parallel \\
 1 & \longrightarrow & \mathbb{Z}(1) & \longrightarrow & \pi_1 \times & \longrightarrow & \pi_1 \times \pi_1 \longrightarrow 1
 \end{array}$$

Then, it follows from various definitions involved that  
 Serre-Hochschild  $\pi_1 \times \pi_1 \rightarrow \pi_1$

$$H^2(\pi_1 \times \pi_1, \mathbb{Z}(1)) \cong H^1(\pi_1, \text{Hom}(\pi_1^{\text{ab}}, \mathbb{Z}(1)))$$



On the other hand, since



We conclude that  $\phi$  coincides

$$\begin{array}{ccc} \pi_1 \twoheadrightarrow \overline{\pi_1}^{ab} \xrightarrow{\sim} \text{Hom}(\overline{\pi_1}^{ab}, \mathbb{Z} \oplus \mathbb{Z}) \\ \uparrow \text{Poincaré duality} \\ \hookrightarrow \text{Hom}(\pi_1, \mathbb{Z} \oplus \mathbb{Z}) \end{array}$$

[cf., e.g., [Milne], ch VI, lem R.2]

$$\begin{aligned} \therefore \ker \rho &= \ker \beta \\ &= \ker (\pi_1 \twoheadrightarrow \overline{\pi_1}^{ab}) // \end{aligned}$$

Prop 6  $\cong \beta_2 : \pi_2^\circ \xrightarrow{\sim} \overline{\pi_2}^\circ : \text{PF-adm}$

Then  $(g^\circ, r^\circ) = (g^\circ, r^\circ)$

☹ Let  $p_i^\square : \overline{\pi_2}^\square \twoheadrightarrow \overline{\pi_1}^\square$  be the natural outer surj induced by

the i-

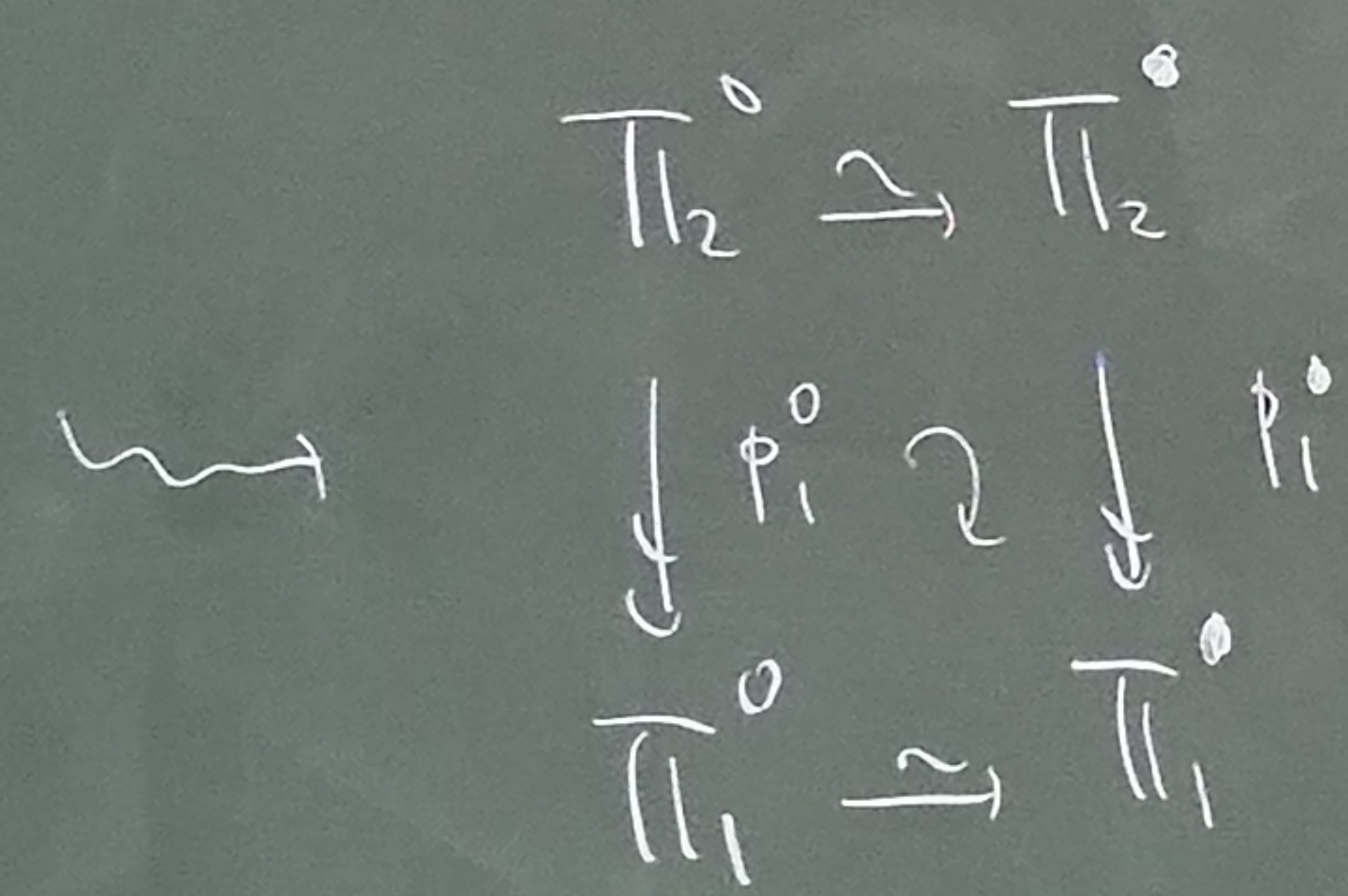
$$\overline{\pi_1}^{ab} //$$

$\pi_2^\circ$  : PF-adm

$$= (g^\circ, h^\circ)$$

$\pi_1^\circ$  be the  
proj induced by

the  $i$ -th proj  $X_2^\square \rightarrow X^\square$   
 To verify props, we may  
 assume WLOG that  
 $\beta(\ker p_i) = \ker p_i^\circ \quad (i=1,2)$



Here note that  
 $r^\circ > 0 \Leftrightarrow \pi_1^\circ$  : free pro-f  
 $\Leftrightarrow \pi_1^\circ$  : free pro-f  
 $\Leftrightarrow r^\circ > 0$

Thus, if  $r^\circ = 0$ , then  $r^\circ = 0$   
 Moreover, in this case  
 $2g^\circ = rk_{2\ell}(\pi_1^\circ)^{ab} = rk_{2\ell}(\pi_1^\circ)^{ab} = 2g^\circ$

$$\therefore g^{\circ} = g^{\bullet} //$$

So we may assume that

$$r^{\circ} > 0 \quad (\Leftrightarrow r^{\bullet} > 0)$$

Then

$$1 \rightarrow \pi_{2,1}^{\circ} \rightarrow \pi_2^{\circ} \xrightarrow{p_1^{\circ}} \pi_1^{\circ} \rightarrow 1$$

$$\begin{array}{ccccccc} & & \downarrow \beta_2 & & \downarrow \beta_1 & & \\ & & \pi_2^{\bullet} & & \pi_1^{\bullet} & & \\ 1 & \rightarrow & \pi_{2,1}^{\bullet} & \rightarrow & \pi_1^{\bullet} & \rightarrow & 1 \end{array}$$

induces

$$\pi_1^{\circ} \xrightarrow{p^{\circ}} \text{Aut}(\pi_{2,1}^{\circ})$$

$$\downarrow \quad \supset \quad \downarrow$$

$$\pi_1^{\bullet} \xrightarrow{p^{\bullet}} \text{Aut}(\pi_{2,1}^{\bullet})$$

$$\therefore \ker(p^{\circ}) \cong \ker(p^{\bullet})$$

$\Rightarrow$   
lem 5

$$\ker(\pi_1^{\circ} \rightarrow (\pi_1^{\circ})^{ab})$$

$$\cong \ker(\pi_1^{\bullet} \rightarrow (\pi_1^{\bullet})^{ab})$$

lem 5 + some argument

theory of the wt filtration  
of  $\Pi_n$

$$\Rightarrow \ker(\pi_1^{\circ ab} \rightarrow \overline{\pi_1^{\circ ab}})$$

$$\xrightarrow{\cong} \ker(\pi_1^{\circ ab} \rightarrow \overline{\pi_1^{\circ ab}})$$

$$\therefore (2g^{\circ} + r^{\circ} + 1) - 2g^{\circ}$$

$$= (2g^{\circ} + r^{\circ} + 1) - 2g^{\circ}$$

$$\therefore r^{\circ} = r^{\circ}$$

$$\therefore g^{\circ} = g^{\circ} //$$

Prop? Recall that  $n \geq 2$

$$\text{Let } \varphi: \Pi_n^{ab} \rightarrow \underbrace{\pi_1^{ab} \times \dots \times \pi_1^{ab}}_m$$

be the surj induced by

$$X_n \hookrightarrow X_{x_1} \times \dots \times X_{x_m}$$